On the Scalar Curvature of Complex Surfaces

Claude LeBrun*
SUNY Stony Brook

December, 1994

Abstract

Let (M, J) be a minimal compact complex surface of Kähler type. It is shown that the smooth 4-manifold M admits a Riemannian metric of positive scalar curvature iff (M, J) admits a *Kähler* metric of positive scalar curvature. This extends previous results of Witten and Kronheimer.

A *complex surface* is a pair (M, J) consisting of a smooth compact 4-manifold M and a complex structure J on M; the latter means an almost-complex structure tensor $J : TM \to TM$, $J^2 = -1$, which is locally isomorphic to the usual constant-coefficient almost-complex structure on $\mathbf{R}^4 = \mathbf{C}^2$. Such a complex surface is called *minimal* if it contains no embedded copy C of S^2 such that $J(TC) = TC$ and such that $C \cdot C = -1$ in homology; this is equivalent to saying that (M, J) cannot be obtained from another complex surface $(\check{M}, \check{J})$ by the procedure of "blowing up a point."

A Riemannian metric g on M is said to be is said to be *Kähler* with respect to J if g is J-invariant and J is parallel with respect to the metric connection of g. If such metrics actually exist, (M, J) is then said to be of *Kähler type*; by a deep result [1] of Kodaira, Todorov, and Siu, this holds iff $b_1(M)$ is even.

The purpose of the present note is to prove the following:

Theorem 1 *Let (M, J) be a minimal complex surface of Kähler type. Then the following are equivalent:*

(a) M admits a Riemannian metric of positive scalar curvature;

(b) (M, J) admits a Kähler metric of positive scalar curvature;

(c) (M, J) is either a ruled surface or $\mathbf{CP}_2$.

*Supported in part by NSF grant DMS-9003263.

Here a minimal complex surface (M, J) is said to be *ruled* iff it is the total space of a holomorphic $\mathbf{CP}_1$-bundle over a compact complex curve.

The equivalence between (b) and (c) was proved in one of Yau's first papers [14]. By contrast, the link between (a) and (c) came to light only recently, when Witten [13] discovered that a Kähler surface with $b^+ > 1$ cannot admit a metric of positive scalar curvature. Kronheimer [6] then used a refinement of Witten's method to prove that a minimal surface of general type cannot admit positive-scalar-curvature metrics. In essence, what will be shown here is simply that Kronheimer's method can, with added care, also be applied to the case of minimal elliptic surfaces.

1 Seiberg-Witten Invariants

The ideas presented in this section are fundamentally due to Witten [13], but much of the formal framework and many technical results are due to Kronheimer-Mrowka [7]. The work of Taubes [12] contains less elementary but more robust proofs of other key results presented here. See also [8].

Let (M, g) be a smooth compact Riemannian 4-manifold, and suppose that M admits an almost-complex structure. Then the given component of the almost-complex structures on M contains almost-complex structures $J : TM \to TM$, $J^2 = -1$ which are compatible with g in the sense that $J^*g = g$. Fixing such a J, the tangent bundle TM of M may be given the structure of a rank-2 complex vector bundle $T^{1,0}$ by defining scalar multiplication by i to be J. Setting $\wedge^{0,p} := \wedge^p \overline{T^{1,0}}^* \cong \wedge^p T^{1,0}$, we may then then define rank-2 complex vector bundles $V_\pm$ on M by

$$\begin{aligned} V_+ &= \wedge^{0,0} \oplus \wedge^{0,2} \\ V_- &= \wedge^{0,1}, \end{aligned} \tag{1}$$

and g induces canonical Hermitian inner products on these bundles.

As described, these bundles depend on the choice of a particular almost-complex structure J, but they have a deeper meaning [3] that depends only on the homotopy class c of J; namely, if we restrict to a contractible open set $U \subset M$, the bundles $V_\pm$ may be canonically identified with $\mathbf{S}_\pm \otimes L^{1/2}$, where $\mathbf{S}_\pm$ are the left- and right-handed spinor bundles of g, and $L^{1/2}$ is a complex line bundle whose square is the 'anti-canonical' line-bundle $L = (\overline{\wedge^{0,2}})^* \cong \wedge^{0,2}$. For each connection A on L compatible with the g-induced inner product, we can thus define a corresponding Dirac operator

$$D_A : C^\infty(V_+) \to C^\infty(V_-).$$

If J is parallel with respect to g, so that (M, g, J) is a Kähler manifold, and if A is the Chern connection on the anti-canonical bundle L, then $D_A = \sqrt{2}(\overline{\partial} \oplus \overline{\partial}^*)$, where $\overline{\partial} : C^\infty(\wedge^{0,0}) \to C^\infty(\wedge^{0,1})$ is the J-antilinear part of the exterior

differential d, acting on complex-valued functions, and where $\overline{\partial}^* : C^\infty(\wedge^{0,2}) \to C^\infty(\wedge^{0,1})$ is the formal adjoint of the map induced by the exterior differential d acting on 1-forms; more generally, D_A will differ from $\sqrt{2}(\overline{\partial} \oplus \overline{\partial}^*)$ by only 0^{th} order terms.

In addition to the metric g and class c of almost-complex structures J, suppose we also choose some $\varepsilon \in C^\infty(\wedge^+)$. The perturbed Seiberg-Witten equations

$$\begin{aligned} D_A\Phi &= 0 & (2) \\ iF_A^+ + \sigma(\Phi) &= \varepsilon & (3) \end{aligned}$$

are then equations for an unknown smooth connection A on L and an unknown smooth section Φ of V_+. Here the purely imaginary 2-form F_A^+ is the self-dual part of the curvature of A, and, in terms of (1), the real-quadratic map $\sigma : V_+ \to \wedge^2_+$ is given by

$$\sigma(f, \phi) = (|f|^2 - |\phi|^2)\frac{\omega}{4} + \Im m(\bar{f}\phi),$$

where $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ is the 'Kähler' form.

For a fixed metric g, let $\mathcal{M}(g)$ denote the set of pairs (A, Φ) which satisfy (2), modulo the action $(A, \Phi) \mapsto (A + 2d\log u, u\Phi)$ of the 'gauge group' of smooth maps $u : M \to S^1 \subset \mathbf{C}$. We may then view (3) as defining a map $\wp : \mathcal{M}(g) \to C^\infty(\wedge^+)$. One can show [7] that this is a proper map, and so, in particular, has compact fibers.

A solution (A, Φ) is called *reducible* if $\Phi \equiv 0$; otherwise, it is called *irreducible*. Let $\mathcal{M}^*(g)$ denote the image in $\mathcal{M}(g)$ of the set of irreducible solutions. Then $\mathcal{M}(g)$ is [7] a smooth Fréchet manifold, and an index calculation shows that the smooth map $\wp : \mathcal{M}^*(g) \to C^\infty(\wedge^+)$ is generically finite-to-one. Let us define a solution (Φ, A) to be *transverse* if it corresponds to a regular point of $\wp$. This holds iff the linearization $C^\infty(V_+ \oplus \wedge^1) \to C^\infty(\wedge^2_+)$ of the left-hand-side of (3), constrained by the linearization of (2), is surjective.

Key Example Let (M, g, J) be a Kähler surface, and let s denote the scalar curvature of g. Set $\varepsilon = (s+1)\omega/4$, set $\Phi = (1, 0) \in \wedge^{0,0} \oplus \wedge^{0,2}$, and let A be the Chern connection on the anti-canonical bundle. Since iF_A is just the Ricci form of (M, g, J), it follows that $iF_A^+ + \sigma(\Phi) = s\omega/4 + \omega/4 = \varepsilon$, and (Φ, A) is thus an irreducible solution of equations (2) and (3).

The linearization of (2) at this solution is just

$$(\overline{\partial} \oplus \overline{\partial}^*)(u + \psi) = -\frac{1}{2}\alpha, \qquad (4)$$

where $(v, \psi) \in C^\infty(V_+)$ is the linearization of $\Phi = (f, \phi)$ and $\alpha \in \wedge^{0,1}$ is the $(0, 1)$-part of the purely imaginary 1-form which is the linearization of A.

Linearizing (3) at our solution yields the operator

$$(v, \psi, \alpha) \mapsto id^+(\alpha - \bar{\alpha}) + \frac{1}{2}(\Re e v)\omega + \Im m\psi.$$

Since the right-hand-side is a real self-dual form, it is completely characterized by its component in the ω direction and its $(0,2)$-part. The ω-component of this operator is just

$$(v, \psi, \alpha) \mapsto \Re e\left[-\bar{\partial}^*\alpha + \frac{v}{2}\right],$$

while the $(0,2)$-component is

$$(v, \psi, \alpha) \mapsto i\bar{\partial}\alpha - \frac{i}{2}\psi.$$

Substituting (4) into these expressions, we obtain the operator

$$\begin{array}{ccc} C^\infty(\mathbf{C} \oplus \wedge^{0,2}) & \longrightarrow & C^\infty(\mathbf{R} \oplus \wedge^{0,2}) \\ (v, \psi) & \mapsto & (\Re e\left[\Delta + \frac{1}{2}\right]v, -i\left[\Delta + \frac{1}{2}\right]\psi), \end{array}$$

which is surjective because $\Delta + \frac{1}{2}$ is a positive self-adjoint elliptic operator. The constructed solution is therefore transverse. □

For any metric g, let c_1^+ denote the image of $c_1(L) \in H^2(M, \mathbf{R})$ under orthogonal[1] projection to the linear subspace $H^+(g)$ of deRham classes which are represented by self-dual 2-forms with respect to g. Given any $\varepsilon \in C^\infty(\wedge^+)$, let ε_H denote its harmonic part; this is a closed self-dual 2-form, since the Laplacian commutes with the Hodge star operator.

Lemma 1 *Let g be any Riemannian metric on M, and let $\varepsilon \in C^\infty(\wedge^+)$. Suppose that $[\varepsilon_H] \neq 2\pi c_1^+$ in deRham cohomology. Then every solution of (2) and (3) is irreducible.*

Proof. If $\Phi \equiv 0$, (3) says that $c_1(L)$ is represented by $\varepsilon/2\pi$ plus an anti-self-dual form. Taking the harmonic part of this representative and projecting to the self-dual harmonic forms then yields $c_1^+ = [\varepsilon_H]/2\pi$. ∎

Definition 1 *If g is a smooth Riemannian metric on M and $\varepsilon \in C^\infty(\wedge^+)$ is such that $[\varepsilon_H] \neq 2\pi c_1^+$, then, with respect to $c = [J]$, we will say that (g, ε) is a* good pair*. The path components of the manifold of all good pairs (g, ε) will be called* chambers*.*

[1] with respect to the intersection form

Lemma 2 *If $b^+(M) > 1$, there is exactly one chamber. If $b^+(M) = 1$, there are exactly two chambers.*

Proof. The projection $(g, \varepsilon) \mapsto g$ factors through the rank-b^+ vector bundle H^+ over the space of Riemannian metrics via a map $(g, \varepsilon) \mapsto (g, [\varepsilon_H])$ with connected fibers. Now $2\pi c_1^+(g)$ is a smooth section of H^+, and since the space of Riemannian metrics is path-connected, the complement of this section is connected if $b^+ > 1$, and has exactly two components if $b^+ = 1$. The result follows. ∎

Lemma 3 *Suppose that $b^+(M) = 1$, $c_1(L) \neq 0$, and $c_1^2(L) \geq 0$. Then $(g, 0)$ is a good pair for any metric g. In particular, the chamber containing $(g, 0)$ is independent of g, and will be called the* preferred chamber.

Proof. Our hypotheses say the intersection form on H^2 is a Lorentzian inner product and that $c_1(L)$ is a non-zero null or time-like vector. Since $2\pi c_1^+(g)$ is the projection of $2\pi c_1(L)$ onto a time-like line, it thus never equals $0 = [0_H]$. ∎

Definition 2 *Let (M, c) be a compact 4-manifold equipped with a homotopy class $c = [J]$ of almost-complex structures. A good pair (g, ε) will be called* excellent *if ε is a regular value of the map $\wp : \mathcal{M}^*(g) \to C^\infty(\wedge^+)$.*

Notice that (g, ε) is excellent iff every solution of (2) and (3) with respect to (g, ε) is irreducible and transverse.

Definition 3 *Let (M, c) be a compact 4-manifold equipped with a class $c = [J]$ of almost-complex structures. If (g, ε) is an excellent pair on M, we define its (mod 2)* Seiberg-Witten invariant *$n_c(M, g, \varepsilon) \in \mathbf{Z}_2$ to be*

$$n_c(M, g, \varepsilon) = \#\{\textit{gauge classes of solutions of (2) and (3)}\} \bmod 2$$

calculated with respect to (g, ε).

Lemma 4 *If two excellent pairs are in the same chamber, they have the same Seiberg-Witten invariant n_c.*

Proof. Any two such pairs can be joined by a path of good pairs which is transverse to $\wp$. This gives a cobordism between the relevant solution spaces. ∎

Definition 4 *Let (M, c) be a smooth 4-manifold equipped with a class of almost-complex structures. If $b^+(M) > 1$, the Seiberg-Witten invariant $n_c(M)$ is defined to be $n_c(M, g, \varepsilon)$, where (g, ε) is any excellent pair. If $b^+(M) = 1$, $c_1(L) \neq 0$, and $c_1^2(L) \geq 0$, then the Seiberg-Witten invariant $n_c(M)$ is defined to be $n_c(M, g, \varepsilon)$, where (g, ε) is any excellent pair in the preferred chamber.*

Theorem 2 *Let (M, J) be a compact complex surface which admits a Kähler metric g; let $c = [J]$. Then there is a chamber for which the Seiberg-Witten invariant n_c is non-zero. Moreover, if $c_1 \cdot [\omega] < 0$, where $[\omega]$ is the Kähler class of g, then the chamber in question contains the good pair $(g, 0)$.*

Proof. Set $\varepsilon = (s+1)\omega/4$, where s is the scalar curvature of the Kähler metric g. Then $\varepsilon_H = (s_0 + 1)\omega/4$, where the average value s_0 of the scalar curvature of g is given by

$$s_0 = \frac{\int_M s\ d\mu}{\int_M d\mu} = \frac{2\int_M \rho \wedge \omega}{\int_M \omega \wedge \omega/2} = 8\pi \frac{c_1 \cdot [\omega]}{[\omega]^2}$$

because the Ricci form ρ represents $2\pi c_1$. On the other hand, $2\pi c_1^+$ is represented by the harmonic form $s_0\omega/4$, so we always have $2\pi c_1^+ \neq [\varepsilon_H]$. This shows that (g, ε) is a good pair. Moreover, if $c_1 \cdot [\omega] < 0$, then $s_0 < 0$, and and $(g, t\varepsilon)$ is a good pair for all $t \in [0, 1]$. Thus it suffices to show that $n_c(M, g, \varepsilon) \equiv 1 \bmod 2$.

We will accomplish this by showing that, with respect to (g, ε) and up to gauge equivalence, there is exactly one solution of the perturbed Seiberg-Witten equations, namely the transverse solution described in the Key Example. Indeed, suppose that $\Phi = (f, \phi)$ is any solution of (2) and (3), and let $\hat{\Phi} = (f, -\phi)$. The Weitzenböck formula for the twisted Dirac operator and equation (3) thus tell us that

$$\begin{aligned} 0 = D_A^* D_A \Phi &= \nabla^*\nabla\Phi + \frac{s}{4}\Phi + \frac{1}{2}F_A \cdot \Phi \\ &= \nabla^*\nabla\Phi + \frac{s}{4}\Phi + \frac{i}{2}\sigma(\Phi) \cdot \Phi - \frac{i}{2}\varepsilon \cdot \hat{\Phi} \\ &= \nabla^*\nabla\Phi + \frac{1}{4}(s + |\Phi|^2)\Phi - \frac{1}{4}(s+1)\hat{\Phi} \end{aligned}$$

because the ± 1-eigenspaces of Clifford multiplication on V_+ by $-2i\omega$ are respectively $\wedge^{0,0}$ and $\wedge^{0,2}$. Projecting to the ∇-invariant sub-bundle $\wedge^{0,0} \subset V_+$ now yields

$$0 = 4\nabla^*\nabla f - f + |\Phi|^2 f.$$

At the maximum of $|f|^2$, we thus have

$$\begin{aligned} 0 \leq 2\Delta\langle f, f\rangle &= 4\langle \Delta f, f\rangle - 4\langle \nabla f, \nabla f\rangle \\ &\leq |f|^2 - |\Phi|^2|f|^2 \\ &\leq (1 - |f|^2)|f|^2 \end{aligned}$$

because $|\Phi|^2 = |f|^2 + |\phi|^2 \geq |f|^2$. This gives us the inequality

$$|f|^2 \leq 1 \tag{5}$$

at all points of M, with equality only when $\phi = 0$ and $\nabla f = 0$.

On the other hand,the closed 2-form iF_A is in the same cohomology class as the Ricci form ρ, which satisfies $\langle \omega, \rho \rangle = s/2$. Writing $iF_A = \rho + d\beta$ for some 1-form β, we have

$$\begin{aligned}
\int_M \left(|f|^2 - |\phi|^2\right) d\mu &= 2\int_M \langle \omega, \sigma(f,\phi)\rangle d\mu \\
&= 2\int_M \langle \omega, -iF_A + \varepsilon\rangle d\mu \\
&= 2\int_M \langle \omega, -\rho - d\beta\rangle d\mu + \int_M \langle \omega, (s+1)\frac{\omega}{2}\rangle d\mu \\
&= -\int_M s\ d\mu - 2\int_M \langle d^*\omega, \beta\rangle d\mu + \int_M (s+1) d\mu \\
&= \int_M 1\ d\mu,
\end{aligned}$$

which is to say that

$$\int_M (|f|^2 - 1) d\mu = \int_M |\phi|^2 d\mu \geq 0.$$

The C^0 estimate (5) thus implies that $|f|^2 \equiv 1$, $\phi \equiv 0$, and $\nabla f \equiv 0$. The connection ∇ induced on the $\wedge^{0,0}$ by A is therefore flat and trivial, and A is thus gauge equivalent to the Chern connection on L. Hence our solution coincides, up to gauge transformation, with that of the example; in particular, every solution with respect to (g, ε) is irreducible and transverse, and (g, ε) is an excellent pair. But since there is only one gauge class of solutions with respect to (g, ε), it follows that $n_c = 1 \bmod 2$ for the chamber containing (g, ε). ∎

Theorem 3 *Let M be a compact 4-manifold which admits a class $c = [J]$ of almost-complex structures and has $b^+ > 0$. If g is a metric of positive scalar curvature on M, then $(g, 0)$ is in the closure of a chamber for which $n_c = 0$.*

Proof. Suppose not. For every $\epsilon > 0$, there is a ε such that $\sup |\varepsilon| < 2\epsilon$ and such that (g, ε) is an excellent pair. If $n_c(M, g, \varepsilon) \neq 0$, there is a solution $\Phi \not\equiv 0$ of equations (2) and (3) with respect to g and $\varepsilon = 0$. The Weitzenböck formula

$$0 = D_A^* D_A \Phi = \nabla^* \nabla \Phi + \frac{s + |\Phi|^2}{4}\Phi - \frac{i}{2}\varepsilon \cdot \Phi$$

then implies that

$$0 > \int_M \left(\frac{s-\epsilon}{4}\right)|\Phi|^2 d\mu.$$

Taking $\epsilon < \min s$ then yields a contradiction. ∎

2 Surface Classification and Scalar Curvature

Recall [1] that the Kodaira dimension $\mathrm{Kod}(M,J) \in \{-\infty, 0, 1, 2\}$ of a compact complex surface (M,J) is defined to be $\limsup(\log h^0(M, \mathcal{O}(L^{*\otimes m}))/\log m)$. The following well-known degree argument may be found e.g. in [14].

Lemma 5 *Let $[\omega]$ be a Kähler class on a compact complex surface (M,J) of Kod ≥ 0. Then $c_1 \cdot [\omega] \leq 0$, with equality iff (M,J) is a minimal surface of Kod $= 0$.*

Proof. If $\mathrm{Kod}(M,J) \geq 0$, some positive power κ^m of the canonical line bundle has a holomorphic section. Let $\mathbf{D}$ be the holomorphic curve, counted with appropriate multiplicity, where this section vanishes. The homology class $[\mathbf{D}] \in H_2(M)$ is then the Poincaré dual of $c_1(\kappa^m) = -mc_1(L)$. The area of $\mathbf{D}$ is thus

$$\int_{\mathbf{D}} \omega = -mc_1 \cdot [\omega]$$

which shows that $c_1 \cdot [\omega] \leq 0$, with equality iff $\mathbf{D} = \emptyset$. Since the latter happens iff κ^m is holomorphically trivial, the result follows. ∎

This leads us directly to a result first discovered by Kronheimer [6].

Proposition 1 *(Kronheimer) Let (M,J) be a minimal complex surface of Kod $= 2$. Then M does not admit a Riemannian metric of positive scalar curvature.*

Proof. Such a surface is automatically [1] of Kähler type, and has $c_1^2 > 0$. The Seiberg-Witten invariant $n_c(M)$ is of (M,J) is thus well-defined by Lemma 3 and is non-zero by virtue of Theorem 2. The result therefore follows by Theorem 3. ∎

Similar reasoning yields

Proposition 2 *Let (M,J) be a minimal complex surface of Kähler type with Kod $= 1$. Then M does not admit a Riemannian metric of positive scalar curvature.*

Proof. Such a surface must [1] have $c_1^2 = 0$ and $c_1 \neq 0$. The Seiberg-Witten invariant $n_c(M)$ of (M, J) is thus well-defined by Lemma 3, and the conclusion now follows by the same argument used above. ∎

The next case is actually covered by existing results [10, 2, 11], but a Seiberg-Witten proof is given for the sake of completeness.

Proposition 3 *Let (M, J) be a minimal complex surface of Kähler type such that Kod$(M, J) = 0$. Then M does not admit a Riemannian metric of positive scalar curvature.*

Proof. Any such an M is finitely covered by a surface $\tilde{M}$ with $b^+ = 3$; in fact, $\tilde{M}$ is either a K3 surface or a 4-torus. The Seiberg-Witten invariant $n_c(\tilde{M})$ is thus well-defined by Lemma 2 and is non-zero by virtue of Theorem 2. By Theorem 3, $\tilde{M}$ does not admit a metric with $s > 0$. The result therefore follows because any metric on M can be pulled back to $\tilde{M}$. ∎

Our next result immediately implies Theorem 1:

Theorem 4 *Let (M, J) be a minimal surface of Kähler type. If M admits a Riemannian metric of positive scalar curvature, then (M, J) is either $\mathbf{CP}_2$ or a ruled surface. As a consequence, (M, J) therefore carries* Kähler *metrics of positive scalar curvature.*

Proof. By the proceeding Propositions, (M, J) must have Kodaira dimension $-\infty$. The Kodaira-Enriques classification [1] thus says that (M, J) is either $\mathbf{CP}_2$ or a ruled surface. Now [14] any minimal ruled surface admits Kähler metrics of positive scalar curvature; indeed, if $M \cong \mathbf{P}(E)$, where $\varpi : E \to C$ is rank-2 holomorphic vector bundle, then, for any Kähler form ω_C on the Riemann surface C and any Hermitian norm $h : E \to \mathbf{R}$ on the complex vector bundle E, the $(1, 1)$-form

$$\omega = \varpi^* \omega_C + \epsilon \left(i \partial \bar{\partial} \log h \right)$$

is a Kähler form on M with positive scalar curvature if $\epsilon > 0$ is sufficiently small. Since the Fubini-Study metric on $\mathbf{CP}_2$ is also a Kähler metric of positive scalar curvature, the result follows. ∎

Since the minimality hypothesis only features as a technicality in connection with the $b^+ = 1$ case, the following conjecture now seems extremely credible:

Conjecture 1 *Let (M, J) be a compact complex surface with b_1 even. Then the following are equivalent:*

(a) M admits a Riemannian metric of positive scalar curvature;

(b) (M, J) *admits a Kähler metric of positive scalar curvature;*

(c) (M, J) *is either* $\mathbf{CP}_2$ *or a blow-up of some minimal ruled surface.*

However, even (b) $\Leftrightarrow$ (c) is only known 'generically;' cf. [4, 5, 9].

References

[1] W. Barth, C. Peters, and A. Van de Ven, **Compact Complex Surfaces**, Springer-Verlag, 1984.

[2] M. Gromov and H.B. Lawson, "Spin and Scalar Curvature in teh Presence of the Fundamental Group," **Ann. Math. 111** (1980) 209–230.

[3] N.J. Hitchin, *Harmonic Spinors*, **Adv. Math. 14** (1974) 1–55.

[4] N.J. Hitchin, *On the Curvature of Rational Surfaces*, **Proc. Symp. Pure Math. 27** (1975) 65–80.

[5] J.-S. Kim, C. LeBrun and M. Pontecorvo, *Scalar-Flat Kähler Surfaces of All Genera*, preprint, 1994.

[6] P. Kronheimer, private communication.

[7] P. Kronheimer and T. Mrowka, *The Genus of Embedded Surfacs in the Complex Projective Plane*, **Math. Res. Lett.** to appear.

[8] C. LeBrun, *Einstein Metrics and Mostow Rigidity*, **Math. Res. Lett.** to appear.

[9] C. LeBrun and S. Simanca, *On Kähler Surfaces of Constant Positive Scalar Curvature*, **J. Geom. Analysis,** to appear.

[10] A. Lichnerowicz, *Spineurs Harmoniques*, **C.R. Acad. Sci. Paris 257** (1963) 7–9.

[11] R. Schoen and S.-T. Yau, *The Structure of Manifolds of Positive Scalar Curvature,* **Man. Math. 28** (1979) 159–183.

[12] C.H. Taubes, *The Seiberg-Witten Invariants and Symplectic Forms*, **Math. Res. Lett.** to appear.

[13] E. Witten, *Monopoles and Four-Manifolds*, preprint, 1994.

[14] S.-T. Yau, *On the Curvature of Compact Hermitian Manifolds,* **Inv. Math. 25** (1974) 213-239.